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New Approach for Solving Master Equations of Density Operators for the Raman-Coupled Model with Cavity Damping

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Abstract By virtue of the thermo entangled state representation, in which one mode is fictitious accompanying the system mode, we exhibit a novel approach to deriving density operator for a Raman-coupled model with damping of the cavity mode. The normal ordering forms of density matrix elements can be obtained, and the corresponding Wigner functions are also derived.

Keywords Thermo entangled state · $SU(1,1)$ Lie Algebra · Raman-coupled model · Wigner function

1 Introduction

It is well known that the phenomenon of damping and decoherence always happen when a system is immersed in reservoir and is described by a master equation [1–4]. Usually, solving master equations needs to use either the Langevin equation or the Fokker-Planck equation [5] after recasting the density operators into some definite representations, e.g., particle number representation (Q-function), coherent state representation (P-representation) [6], or the Wigner representation [7, 8]. Here enlighten by [9, 10], we alternatively treat such equation by virtue of the newly developed thermo entangled state (TES) [11, 12] defined

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in a two-mode Fock space, in which one mode is a fictitious one representing the effect of environment. It is Takahashi-Umezawa [13, 14] who first introduced the fictitious Fock space to treat ensemble average as a pure state average, this pure state (thermo vacuum state) is also a two-mode squeezed state.

In recent years, much attention has been focused on studying Raman-coupled (RC) model [15–17], in which a single atom interacts with a single cavity mode by Raman-type transitions. If damping is contained in Raman-coupled model, one has to solve the time evolution equation for the density operator ($\hbar = 1$) [18]

$$\dot{\rho} = -i\left[H, \rho\right] + \kappa L_{ir}\left[\rho\right], \tag{1}$$

where

$$H = w_c a^{\dagger} a + g a^{\dagger} a \left(\sigma^{+} + \sigma^{-}\right), \tag{2}$$

κ is the damping constant, and $L_{ir}[\rho]$ takes the form

$$\begin{aligned} L_{ir}\left[\rho\right] &= 2a\rho a^{\dagger} - \rho a^{\dagger} a - a^{\dagger} a\rho + 2n_{th}\left[\left[a, \rho\right], a^{\dagger}\right] \\ &= \left(n_{th}+1\right)\left(2a\rho a^{\dagger} - \rho a^{\dagger} a - a^{\dagger} a\rho\right) + n_{th}\left(2a^{\dagger}\rho a - \rho a a^{\dagger} - a a^{\dagger}\rho\right), \end{aligned} \tag{3}$$

and $n_{th} = 1/\{\exp[w_c/(kT) - 1]\}$ is the number of thermal quanta, k is the Boltzmann constant, and T is the system temperature. The main purpose of this paper is to investigate and show how the density operator evolves in the Raman-coupled model with finite cavity damping in (1) with the help of the TES and give the Wigner functions for the corresponding density operators. In the following section, we first briefly review the TES $|\eta\rangle$ and its some important properties. In Sect. 3, the equations for the atomic matrix elements of the density operator are decoupled by appropriate linear combinations. Then, the equations of motion of the density operators can be solved analytically by virtue of the TES $|\eta\rangle$ and the disentangling theorem for $SU(1,1)$ Lie Algebra, which shows the origin of density operator's time-evolution. We devote Sect. 4 to deriving the normal ordering forms of density matrix elements and the Wigner functions for given initial state.

2 Thermo Entangled State $|\eta\rangle$

Takahashi and UmezawaIn in [13] introduced thermo Field Dynamics (TFD) to convert the statistical average at nonzero temperature T into equivalent pure state expectation value at the expense of introducing a fictitious field (i.e., so-called tilde-conjugate field). Thus every state $|n\rangle$ in the original real field space $\mathcal{H}$ is accompanied by a corresponding state $|\tilde{n}\rangle$ in $\tilde{\mathcal{H}}$. A similar rule holds for operators: every operator a acting on $\mathcal{H}$ has an image $\tilde{a}$ acting on $\tilde{\mathcal{H}}$. For a harmonic oscillator system,

$$|0(\beta)\rangle = \exp\left(a^{\dagger}\tilde{a}^{\dagger}\tanh\theta\right)|0, \tilde{0}\rangle = S(\theta)|0, \tilde{0}\rangle, \tag{4}$$

where

$$S(\theta) \equiv \exp\left[-\theta\left(a\tilde{a} - a^{\dagger}\tilde{a}^{\dagger}\right)\right] \tag{5}$$

is the thermo squeezing operator which transforms the zero-temperature vacuum $|0, \tilde{0}\rangle$ into the thermo vacuum state $|0(\beta)\rangle$, and $\beta = 1/kT$, k is the Boltzmann constant, T is the system temperature; θ is related to the Bose distribution by

$$\tanh\theta = \exp\left(-\frac{\hbar\omega}{2kT}\right). \tag{6}$$

At finite temperature, according to TFD, the number state $|n\rangle$ is replaced by $|n,\tilde{n}\rangle$. We have introduced the coherent thermo state representation $|\eta\rangle$ [11, 12],

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta a^\dagger - \eta^*\tilde{a}^\dagger + a^\dagger\tilde{a}^\dagger\right)|0,\tilde{0}\rangle, \quad \eta = \eta_1 + i\eta_2, \tag{7}$$

where $\tilde{a}^\dagger$ is a fictitious mode (tilde mode) accompanying $a^\dagger$, introduced for convenience. $|\tilde{0}\rangle$ is annihilated by $\tilde{a}$, $[\tilde{a},\tilde{a}^\dagger] = 1$.The states $|\eta\rangle$ make up an orthogonal-complete relation

$$\int \frac{d^2\eta}{\pi}|\eta\rangle\langle\eta| = 1, \qquad \langle\eta'|\eta\rangle = \pi\delta^{(2)}\left(\eta'-\eta\right), \tag{8}$$

and obey the following eigenvector equations

$$\left(a-\tilde{a}^\dagger\right)|\eta\rangle = \eta|\eta\rangle, \qquad \left(\tilde{a}-a^\dagger\right)|\eta\rangle = -\eta^*|\eta\rangle. \tag{9}$$

When $\eta = 0$, it then follows

$$|\eta = 0\rangle = \exp\left(a^\dagger\tilde{a}^\dagger\right) = \sum_{n=0}^{\infty}|n,\tilde{n}\rangle \equiv |\boldsymbol{I}\rangle. \tag{10}$$

It is obtained that

$$\begin{aligned} a|\boldsymbol{I}\rangle &= \tilde{a}^\dagger|\boldsymbol{I}\rangle, \qquad a^\dagger|\boldsymbol{I}\rangle = \tilde{a}|\boldsymbol{I}\rangle, \\ \left(a^\dagger a\right)^n|\boldsymbol{I}\rangle &= \left(\tilde{a}^\dagger\tilde{a}\right)^n|\boldsymbol{I}\rangle, \qquad \left(aa^\dagger\right)^n|\boldsymbol{I}\rangle = \left(\tilde{a}\tilde{a}^\dagger\right)^n|\boldsymbol{I}\rangle. \end{aligned} \tag{11}$$

Note that (11) is an important property for the following derivation.

3 Time Evolution of $|\rho(t)\rangle$ in the RC Model with Cavity Damping

In order to further conveniently study the equation of motion for density operator in (1), we firstly describe the above system in the interaction picture. By appealing the unitary operator

$$U(t) = \exp\left(-iw_c a^\dagger at\right) \tag{12}$$

to all the operators in (1)–(3), due to

$$U^{-1}\rho U \Longrightarrow \rho, \qquad U^{-1}HU \Longrightarrow H, \qquad \ldots, \tag{13}$$

we derive the master equation of operator in the interaction picture as follows

$$\dot{\rho} = -ig\left[a^\dagger a\left(\sigma^+ + \sigma^-\right),\rho\right] + \kappa L_{ir}[\rho]. \tag{14}$$

Using the Pauli spin matrices $\sigma_x = \sigma^+ + \sigma^- = \sigma_1, \sigma_y = \sigma_2, \sigma_z = \sigma_3$, and the unit matrix $\sigma_0 = \mathbf{1}$, we may rewrite the density operator in terms of ρ_i, $i = 0, 1, 2, 3$,

$$\rho = \frac{1}{2}\sum_{i=0}^{3}\rho_i\sigma_i = \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix}. \tag{15}$$

Note that

$$\rho_{\uparrow\uparrow} = \langle\uparrow|\rho|\uparrow\rangle, \qquad \rho_{\uparrow\downarrow} = \langle\uparrow|\rho|\downarrow\rangle, \qquad \rho_{\downarrow\uparrow} = \langle\downarrow|\rho|\uparrow\rangle, \qquad \rho_{\downarrow\downarrow} = \langle\downarrow|\rho|\downarrow\rangle, \tag{16}$$

and

$$\begin{aligned} &\rho_0 = \rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow}, \qquad \rho_1 = \rho_{\uparrow\downarrow} + \rho_{\downarrow\uparrow}, \\ &\rho_2 = i\left(\rho_{\uparrow\downarrow} - \rho_{\downarrow\uparrow}\right), \qquad \rho_3 = \rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow}. \end{aligned} \tag{17}$$

According to the above equations, (14) can be split into the following four equations

$$\begin{aligned} \dot{\rho}_0 &= -ig\left[a^\dagger a, \rho_1\right] + \kappa L_{ir}[\rho_0], \\ \dot{\rho}_1 &= -ig\left[a^\dagger a, \rho_0\right] + \kappa L_{ir}[\rho_1], \\ \dot{\rho}_2 &= -g\left(a^\dagger a\rho_3 + \rho_3 a^\dagger a\right) + \kappa L_{ir}[\rho_2], \\ \dot{\rho}_3 &= g\left(a^\dagger a\rho_2 + \rho_2 a^\dagger a\right) + \kappa L_{ir}[\rho_3]. \end{aligned} \tag{18}$$

It is seen that only the first two and the last two equations are coupled, respectively. By taking

$$\rho_\pm = \rho_0 \pm \rho_1, \tag{19}$$

the first two equations can be decoupled as

$$\dot{\rho}_\pm = \pm ig\left[\rho_\pm, a^\dagger a\right] + \kappa L_{ir}\left[\rho_\pm\right]. \tag{20}$$

The other two equations can be cast into one equation as well. By letting

$$\rho_c = \rho_3 + i\rho_2, \tag{21}$$

the last two equations are equivalent to the single equation

$$\dot{\rho}_c = -ig\left(a^\dagger a\rho_c + \rho_c a^\dagger a\right) + \kappa L_{ir}[\rho_c]. \tag{22}$$

At this point, solving the equation of motion in (14) is equivalent to solving (20) and (22).

Next, we would like to resolve the master equations in (20) and (22) by virtue of the TES $|\eta\rangle$. Multiplying (20) from the right by $|\boldsymbol{I}\rangle$ and using (3) and (11), we have

$$\begin{aligned} \frac{\partial}{\partial t}|\rho_\pm\rangle &= \pm ig\left(\rho_\pm a^\dagger a - a^\dagger a\rho_\pm\right)|\boldsymbol{I}\rangle + \kappa L_{ir}\left[\rho_\pm\right]|\boldsymbol{I}\rangle \\ &= \pm ig\left(\tilde{a}^\dagger\tilde{a} - a^\dagger a\right)|\rho_\pm\rangle \\ &\quad + \kappa\left(n_{th}+1\right)\left(2a\tilde{a} - \tilde{a}^\dagger\tilde{a} - a^\dagger a\right)|\rho_\pm\rangle + \kappa n_{th}\left(2a^\dagger\tilde{a}^\dagger - \tilde{a}\tilde{a}^\dagger - aa^\dagger\right)|\rho_\pm\rangle, \end{aligned} \tag{23}$$

where $|\rho_\pm\rangle \equiv \rho_\pm|\boldsymbol{I}\rangle$. To deal with the above equations, we set

$$\begin{aligned} K_+ &= a^\dagger \tilde{a}^\dagger, \qquad K_- = a\tilde{a}, \\ K_3 &= \frac{a^\dagger a + \tilde{a}^\dagger \tilde{a} + 1}{2}, \end{aligned} \tag{24}$$

which satisfy the $SU(1,1)$ Lie Algebra [19, 20]

$$\left[K_-, K_+\right] = 2K_3, \qquad \left[K_3, K_\pm\right] = \pm K_\pm, \tag{25}$$

with $K_0 = a^\dagger a - \tilde{a}^\dagger \tilde{a}$ serving as the Casimir operator. Thus, we may derive the formal solution of (23) as

$$|\rho_\pm\rangle = \exp\left(\mp igtK_0 + \kappa t\right)\exp\left(\kappa_+ K_+ + \kappa_3 K_3 + \kappa_- K_-\right)|\rho_\pm(0)\rangle\,, \tag{26}$$

where

$$\begin{aligned} \kappa_+ &= 2\kappa n_{th} t, \\ \kappa_3 &= -2\kappa\left(2n_{th}+1\right)t, \\ \kappa_- &= 2\kappa\left(n_{th}+1\right)t \end{aligned} \tag{27}$$

and $|\rho_\pm(0)\rangle \equiv \rho_\pm(0)|\boldsymbol{I}\rangle$ with $\rho_\pm(0)$ being the initial density operator with respect to time $t = 0$. Using the disentangling theorem for $SU(1,1)$ [21–23]

$$\begin{aligned} &\exp\left(\kappa_+ K_+ + \kappa_3 K_3 + \kappa_- K_-\right) \\ &\quad = \exp\left(\Gamma_+ K_+\right)\exp\left[\left(2\ln\sqrt{\Gamma_3}\right)K_3\right]\exp\left(\Gamma_- K_-\right), \end{aligned} \tag{28}$$

where

$$\begin{aligned} \Gamma_\pm &= \frac{2\kappa_\pm \sinh\varphi}{2\varphi\cosh\varphi - \kappa_3\sinh\varphi}, \\ \sqrt{\Gamma_3} &= \frac{2\varphi}{2\varphi\cosh\varphi - \kappa_3\sinh\varphi}, \\ \varphi^2 &= \left(\frac{\kappa_3}{2}\right)^2 - \kappa_+\kappa_-. \end{aligned} \tag{29}$$

From (24) and (28), (26) is rewritten as

$$\begin{aligned} |\rho_\pm\rangle &= \exp\left[\mp igt\left(a^\dagger a - \tilde{a}^\dagger\tilde{a}\right) + \kappa t\right]\exp\left(\Gamma_+ a^\dagger\tilde{a}^\dagger\right) \\ &\quad \times \exp\left[\left(2\ln\sqrt{\Gamma_3}\right)\frac{a^\dagger a + \tilde{a}^\dagger\tilde{a} + 1}{2}\right]\exp\left(\Gamma_- a\tilde{a}\right)\rho_\pm(0)|\boldsymbol{I}\rangle \\ &= \exp\left[\mp igt\left(a^\dagger a - \tilde{a}^\dagger\tilde{a}\right) + \kappa t\right]\exp\left(\Gamma_+ a^\dagger\tilde{a}^\dagger\right) \\ &\quad \times \exp\left[\left(\ln\sqrt{\Gamma_3}\right)\left(a^\dagger a + \tilde{a}^\dagger\tilde{a} + 1\right)\right]\exp\left(\Gamma_- a\tilde{a}\right)\rho_\pm(0)|\boldsymbol{I}\rangle \\ &= \sqrt{\Gamma_3}\exp\left[\mp igt\left(a^\dagger a\right) + \kappa t\right] \end{aligned}$$

$$\times \sum_{m,n=0}^{\infty} \frac{\left(\Gamma_{+}a^{\dagger}\right)^{m}}{m!} \exp\left[\left(\ln\sqrt{\Gamma_3}\right)\left(a^{\dagger}a\right)\right] \frac{\left(\Gamma_{-}a\right)^{n}}{n!} \rho_{\pm}\left(0\right)\left(a^{\dagger}\right)^{n}$$
$$\times \exp\left[\left(\ln\sqrt{\Gamma_3}\right)\left(a^{\dagger}a\right)\right] a^{m} \exp\left[\pm igt\left(a^{\dagger}a\right)\right]\left|\boldsymbol{I}\right\rangle, \tag{30}$$

which directly leads to

$$\rho_{\pm} = \sqrt{\Gamma_3}\exp\left[\mp igt\left(a^{\dagger}a\right)+\kappa t\right] \sum_{m,n=0}^{\infty} \frac{\left(\Gamma_{+}a^{\dagger}\right)^{m}}{m!} \exp\left[\left(\ln\sqrt{\Gamma_3}\right)\left(a^{\dagger}a\right)\right] \frac{\left(\Gamma_{-}a\right)^{n}}{n!} \rho_{\pm}\left(0\right)$$
$$\times \left(a^{\dagger}\right)^{n} \exp\left[\left(\ln\sqrt{\Gamma_3}\right)\left(a^{\dagger}a\right)\right] a^{m} \exp\left[\pm igt\left(a^{\dagger}a\right)\right]$$
$$= \sqrt{\Gamma_3}\exp\left(\kappa t\right) \sum_{m,n=0}^{\infty} \frac{\left(\Gamma_{+}\right)^{m}}{m!} \frac{\left(\Gamma_{-}\right)^{n}}{n!} \left(\rho_{\pm}\right)_{mn}, \tag{31}$$

where

$$\left(\rho_{\pm}\right)_{mn} = \exp\left[\mp igt\left(a^{\dagger}a\right)\right]\left(a^{\dagger}\right)^{m} \exp\left[\left(\ln\sqrt{\Gamma_3}\right)\left(a^{\dagger}a\right)\right] a^{n} \rho_{\pm}\left(0\right)$$
$$\times \left(a^{\dagger}\right)^{n} \exp\left[\left(\ln\sqrt{\Gamma_3}\right)\left(a^{\dagger}a\right)\right] a^{m} \exp\left[\pm igt\left(a^{\dagger}a\right)\right]. \tag{32}$$

Letting

$$M_{\pm mn} = \left[\sqrt{\Gamma_3}\exp\left(\kappa t\right) \frac{\left(\Gamma_{+}\right)^{m}}{m!} \frac{\left(\Gamma_{-}\right)^{n}}{n!}\right]^{1/2}$$
$$\times \exp\left[\mp igt\left(a^{\dagger}a\right)\right]\left(a^{\dagger}\right)^{m} \exp\left[\left(\ln\sqrt{\Gamma_3}\right)\left(a^{\dagger}a\right)\right] a^{n}, \tag{33}$$

(32) can be further rewritten as

$$\rho_{\pm} = \sum_{m,n=0}^{\infty} M_{\pm mn}\rho_{\pm}\left(0\right) M_{\pm mn}^{\dagger}, \tag{34}$$

so $M_{\pm mn}$ is identified as the Kraus operator [24].

Similarly, the explicit form of evolution of the density operator ρ_c in (22) can also be derived. By operating the ket $|\boldsymbol{I}\rangle$ from the right-hand side of (22), we have

$$\frac{\partial}{\partial t}\left|\rho_c\right\rangle = -ig\left(a^{\dagger}a\rho_c + \rho_c a^{\dagger}a\right)\left|\boldsymbol{I}\right\rangle + \kappa\left(n_{th}+1\right)\left(2a\tilde{a} - \tilde{a}^{\dagger}\tilde{a} - a^{\dagger}a\right)\left|\rho_c\right\rangle$$
$$+ \kappa n_{th}\left(2a^{\dagger}\tilde{a}^{\dagger} - \tilde{a}\tilde{a}^{\dagger} - aa^{\dagger}\right)\left|\rho_c\right\rangle$$
$$= -ig\left(a^{\dagger}a + \tilde{a}^{\dagger}\tilde{a}\right)\left|\rho_c\right\rangle + \kappa\left(n_{th}+1\right)\left(2a\tilde{a} - \tilde{a}^{\dagger}\tilde{a} - a^{\dagger}a\right)\left|\rho_c\right\rangle$$
$$+ \kappa n_{th}\left(2a^{\dagger}\tilde{a}^{\dagger} - \tilde{a}\tilde{a}^{\dagger} - aa^{\dagger}\right)\left|\rho_c\right\rangle$$
$$= \left(-2ig\frac{a^{\dagger}a + \tilde{a}^{\dagger}\tilde{a} + 1}{2} + ig\right)\left|\rho_c\right\rangle$$
$$+ \left[2\kappa n_{th}\left(a^{\dagger}\tilde{a}^{\dagger}\right) - 2\kappa\left(2n_{th}+1\right)\left(\frac{a^{\dagger}a + \tilde{a}^{\dagger}\tilde{a} + 1}{2}\right)\right.$$

$$+2\kappa\left(n_{th}+1\right)(a\tilde{a})+\kappa\Bigg]|\rho_c\rangle$$

$$=\left[(ig+\kappa)+2\kappa n_{th}\left(a^{\dagger}\tilde{a}^{\dagger}\right)+2\kappa\left(n_{th}+1\right)(a\tilde{a})\right.$$

$$\left.-2\left(2\kappa n_{th}+\kappa+ig\right)\left(\frac{a^{\dagger}a+\tilde{a}^{\dagger}\tilde{a}+1}{2}\right)\right]|\rho_c\rangle, \tag{35}$$

whose formal solution is

$$|\rho_c\rangle=\exp\left[(ig+\kappa)t\right]\exp\left[\left(\kappa'_{+}K_{+}+\kappa'_{-}K_{-}+\kappa'_{3}K_{3}\right)\right]|\rho_c(0)\rangle, \tag{36}$$

where

$$\begin{aligned}\kappa'_{+}&=2\kappa n_{th}t,\\ \kappa'_{-}&=2\kappa\left(n_{th}+1\right)t,\\ \kappa'_{3}&=-2\left(2\kappa n_{th}+\kappa+ig\right)t,\end{aligned} \tag{37}$$

and (24) has been used. In the same way, the disentangled form of (36) is described as

$$|\rho_c\rangle=\exp\left[(ig+\kappa)t\right]\exp\left(\Gamma'_{+}K_{+}\right)\exp\left[\left(2\ln\sqrt{\Gamma'_3}\right)K_3\right]\exp\left(\Gamma'_{-}K_{-}\right)|\rho_c(0)\rangle, \tag{38}$$

here

$$\begin{aligned}\Gamma'_{\pm}&=\frac{2\kappa'_{\pm}\sinh\varphi'}{2\varphi'\cosh\varphi'-\kappa'_3\sinh\varphi'},\\ \sqrt{\Gamma'_3}&=\frac{2\varphi'}{2\varphi'\cosh\varphi'-\kappa'_3\sinh\varphi'},\\ \varphi'^2&=\left(\frac{\kappa'_3}{2}\right)^2-\kappa'_{+}\kappa'_{-}.\end{aligned} \tag{39}$$

Inserting (24) into (38), we derive

$$|\rho_c\rangle=e^{(ig+\kappa)t}\exp\left(\Gamma'_{+}a^{\dagger}\tilde{a}^{\dagger}\right)\exp\left[\left(2\ln\sqrt{\Gamma'_3}\right)\frac{a^{\dagger}a+\tilde{a}^{\dagger}\tilde{a}+1}{2}\right]\exp\left(\Gamma'_{-}a\tilde{a}\right)\rho_c(0)|\boldsymbol{I}\rangle$$

$$=\sqrt{\Gamma'_3}e^{(ig+\kappa)t}\exp\left(\Gamma'_{+}a^{\dagger}\tilde{a}^{\dagger}\right)\exp\left[\left(\ln\sqrt{\Gamma'_3}\right)\left(a^{\dagger}a+\tilde{a}^{\dagger}\tilde{a}\right)\right]\exp\left(\Gamma'_{-}a\tilde{a}\right)\rho_c(0)|\boldsymbol{I}\rangle$$

$$=\sqrt{\Gamma'_3}e^{(ig+\kappa)t}\sum_{m,n=0}^{\infty}\frac{\left(\Gamma'_{+}a^{\dagger}\right)^m}{m!}\exp\left[\left(\ln\sqrt{\Gamma'_3}\right)\left(a^{\dagger}a\right)\right]\frac{\left(\Gamma'_{-}a\right)^n}{n!}\rho_c(0)$$

$$\times\left(a^{\dagger}\right)^n\exp\left[\left(\ln\sqrt{\Gamma'_3}\right)\left(a^{\dagger}a\right)\right]a^m|\boldsymbol{I}\rangle, \tag{40}$$

which indicates that

$$\rho_c=\sqrt{\Gamma'_3}e^{(ig+\kappa)t}\sum_{m,n=0}^{\infty}\frac{\left(\Gamma'_{+}a^{\dagger}\right)^m}{m!}\exp\left[\left(\ln\sqrt{\Gamma'_3}\right)\left(a^{\dagger}a\right)\right]\frac{\left(\Gamma'_{-}a\right)^n}{n!}\rho_c(0)$$

$$\times \left(a^{\dagger}\right)^{n} \exp\left[\left(\ln\sqrt{\Gamma_3'}\right)\left(a^{\dagger}a\right)\right] a^{m}$$

$$= \sqrt{\Gamma_3'} e^{(ig+\kappa)t} \sum_{m,n=0}^{\infty} \frac{(\Gamma_+')^m}{m!} \frac{(\Gamma_-')^n}{n!} (\rho_c)_{mn}, \tag{41}$$

where

$$(\rho_c)_{mn} = \left(a^{\dagger}\right)^{m} \exp\left[\left(\ln\sqrt{\Gamma_3'}\right)\left(a^{\dagger}a\right)\right] a^{n} \rho_c(0) \left(a^{\dagger}\right)^{n}$$
$$\times \exp\left[\left(\ln\sqrt{\Gamma_3'}\right)\left(a^{\dagger}a\right)\right] a^{m}. \tag{42}$$

4 Wigner Functions for Density Operators $\rho_{\pm}$ and ρ_c

In this section, based on the above discussion, we calculate the Wigner functions of density operators $\rho_{\pm}$ and ρ_c for the given initial state. Usually, the atom is initially considered in one of its degenerate states $|\uparrow\rangle, |\downarrow\rangle$ and the cavity mode is in a coherent state $|\alpha_0\rangle$. Here suppose that the atom is initially in the state $|\uparrow\rangle$, thus the initial condition for the density operator reads

$$\rho_0 = |\alpha_0\rangle\langle\alpha_0| \otimes |\uparrow\rangle\langle\uparrow|. \tag{43}$$

Instituting (43) into (19) and (21), we may easily obtain

$$\rho_{\pm}(0) = \rho_c(0) = |\alpha_0\rangle\langle\alpha_0|. \tag{44}$$

We now turn to give the normal ordering forms for the two density matrix elements, it would bring great convenience for deriving the Wigner functions. Due to $|\alpha_0\rangle = e^{-|\alpha_0|^2+\alpha_0 a^{\dagger}}|0\rangle$ and $|0\rangle\langle 0| =: e^{-a^{\dagger}a}:$, the normal ordering forms of $(\rho_{\pm})_{mn}$ and $(\rho_c)_{mn}$ are expressed as

$$(\rho_{\pm})_{mn} = |\alpha_0|^{2n} e^{-|\alpha_0|^2} \exp(\mp imgt) \left(a^{\dagger}\right)^{m}$$
$$\times \exp\left[\alpha_0 a^{\dagger} e^{\left(\ln\sqrt{\Gamma_3}\mp igt\right)}\right] |0\rangle\langle 0| \exp\left[\alpha_0^* a e^{\left(\ln\sqrt{\Gamma_3}\pm igt\right)}\right] \exp(\pm imgt) a^{m}$$
$$= |\alpha_0|^{2n} e^{-|\alpha_0|^2} \left(a^{\dagger}\right)^{m} \exp\left[\alpha_0 a^{\dagger} e^{\left(\ln\sqrt{\Gamma_3}\mp igt\right)}\right] |0\rangle\langle 0| \exp\left[\alpha_0^* a e^{\left(\ln\sqrt{\Gamma_3}\pm igt\right)}\right] a^{m}$$
$$= |\alpha_0|^{2n} e^{-|\alpha_0|^2} : \left(a^{\dagger}\right)^{m} \exp\left[\alpha_0 a^{\dagger} e^{\left(\ln\sqrt{\Gamma_3}\mp igt\right)}\right]$$
$$\times \exp\left(-a^{\dagger}a\right) \exp\left[\alpha_0^* a e^{\left(\ln\sqrt{\Gamma_3}\pm igt\right)}\right] a^{m} : , \tag{45}$$

and

$$(\rho_c)_{mn} = |\alpha_0|^{2n} e^{-|\alpha_0|^2} \times \left(a^{\dagger}\right)^{m} \exp\left[\left(\ln\sqrt{\Gamma_3'}\right)\left(a^{\dagger}a\right)\right]$$
$$\times \exp\left(\alpha_0 a^{\dagger}\right) |0\rangle\langle 0| \exp\left(\alpha_0^* a\right) \exp\left[\left(\ln\sqrt{\Gamma_3'}\right)\left(a^{\dagger}a\right)\right] a^{m}$$
$$= |\alpha_0|^{2n} e^{-|\alpha_0|^2} \times \left(a^{\dagger}\right)^{m} \exp\left[\left(\alpha_0 a^{\dagger}\right)\left(\ln\sqrt{\Gamma_3'}\right)\right] |0\rangle\langle 0| \exp\left[\alpha_0^* a \left(\ln\sqrt{\Gamma_3'}\right)\right] a^{m}$$

$$= |\alpha_0|^{2n} e^{-|\alpha_0|^2} \colon \left(a^\dagger\right)^m \exp\left[\left(\alpha_0 a^\dagger\right)\left(\ln\sqrt{\Gamma_3'}\right)\right] \exp\left(-a^\dagger a\right)$$
$$\times \exp\left[\alpha_0^* a \left(\ln\sqrt{\Gamma_3'}\right)\right] a^m \colon , \tag{46}$$

respectively. Note that the following operator identities have been appealed

$$\exp\left(x a^\dagger a\right) a \exp\left(-x a^\dagger a\right) = a \exp(-x),$$
$$\exp\left(x a^\dagger a\right) a^\dagger \exp\left(-x a^\dagger a\right) = a^\dagger \exp(x). \tag{47}$$

The Wigner function for density operator is defined as

$$W\left(\alpha, \alpha^*\right) = \mathrm{Tr}\left[\rho \Delta\left(\alpha, \alpha^*\right)\right], \tag{48}$$

where $\triangle(\alpha, \alpha^*)$ is so-called Wigner operator, and has the form

$$\Delta\left(\alpha, \alpha^*\right) = e^{2|\alpha|^2} \int \frac{d^2 z}{\pi^2} |z\rangle \langle -z| \, e^{-2\left(z\alpha^* - z^*\alpha\right)}. \tag{49}$$

Thus, by substituting (45) and (49) into (48) one can deduce

$$W_{\pm mn}\left(\alpha, \alpha^*\right) = |\alpha_0|^{2n} e^{2|\alpha|^2 - |\alpha_0|^2} \int \frac{d^2 z}{\pi^2} \langle -z| \colon \left(a^\dagger\right)^m \exp\left[\alpha_0 a^\dagger e^{\left(\ln\sqrt{\Gamma_3} \mp igt\right)}\right]$$
$$\times \exp\left(-a^\dagger a\right) \exp\left[\alpha_0^* a e^{\left(\ln\sqrt{\Gamma_3} \pm igt\right)}\right] a^m \colon |z\rangle \, e^{-2\left(z\alpha^* - z^*\alpha\right)}$$
$$= |\alpha_0|^{2n} e^{2|\alpha|^2 - |\alpha_0|^2} \int \frac{d^2 z}{\pi^2} \left(-z^*\right)^m z^m e^{-Az^*} e^{A^* z} \langle -z| \, 0\rangle \langle 0| \, z\rangle \, e^{-2\left(z\alpha^* - z^*\alpha\right)}$$
$$= |\alpha_0|^{2n} e^{2|\alpha|^2 - |\alpha_0|^2} \int \frac{d^2 z}{\pi^2} (-1)^m z^{*m} z^m e^{-|z|^2 + \left(A^* - 2\alpha^*\right) z - (A - 2\alpha) z^*}$$
$$= |\alpha_0|^{2n} e^{2|\alpha|^2 - |\alpha_0|^2 - |2\alpha - A|^2} H_{m,m}\left(A^* - 2\alpha^*, A - 2\alpha\right), \tag{50}$$

where $A \equiv \alpha_0 e^{(\ln\sqrt{\Gamma_3} \mp igt)}$ and $H_{m,m}(x, y)$ is the two-variable Hermite polynomial [25], whose generating function is

$$H_{m,n}(x, y) = \frac{\partial^{m+n}}{\partial t^m \partial t'^n} \exp\left(-tt' + tx + t'y\right)|_{t=t'=0}, \tag{51}$$

and the following integral formula

$$H_{m,n}(\xi, \eta) = (-1)^n e^{\xi\eta} \int \frac{d^2 z}{\pi} z^n z^{*m} \exp\left[-|z|^2 + \xi z - \eta z^*\right] \tag{52}$$

has been used in (50). Further, the Wigner functions for the density operators $\rho_\pm$ in (31) are

$$W_\pm\left(\alpha, \alpha^*\right) = \sqrt{\Gamma_3} e^{\kappa t} \sum_{m,n=0}^{\infty} \frac{(\Gamma_+)^m (\Gamma_-)^n}{m! n!} W_{\pm mn}\left(\alpha, \alpha^*\right)$$
$$= \sqrt{\Gamma_3} e^{\kappa t + \Gamma_- |\alpha_0|^2 - |\alpha_0|^2 + 2|\alpha|^2 - |2\alpha - A|^2} \sum_{m=0}^{\infty} \frac{(\Gamma_+)^m}{m!} H_{m,m}\left(A^* - 2\alpha^*, A - 2\alpha\right)$$

$$= \sqrt{\Gamma_3} e^{\kappa t + (\Gamma_- - 1)|\alpha_0|^2 - 2|\alpha|^2 + 2(\alpha A^* + \alpha^* A) - |A|^2} \sum_{m=0}^{\infty} (-\Gamma_+)^m L_m\left(|2\alpha - A|^2\right), \quad (53)$$

where $L_m(x)$ is the mth-order Laguerre polynomials

$$L_m(x) = \sum_{l=0}^{m} \binom{m}{l} \frac{(-x)^l}{l!}, \quad (54)$$

and its relation to the two-variable Hermite polynomial $H_{m,m}(\xi, \eta)$ is

$$L_m(\xi\eta) = \frac{1}{m!}(-1)^m H_{m,m}(\xi, \eta). \quad (55)$$

In the same method, we have

$$\begin{aligned} W_{cmn}\left(\alpha, \alpha^*\right) &= |\alpha_0|^{2n} e^{2|\alpha|^2 - |\alpha_0|^2} \int \frac{d^2 z}{\pi^2} \langle -z| : \left(a^\dagger\right)^m \exp\left[\left(\alpha_0 a^\dagger\right)\left(\ln\sqrt{\Gamma_3'}\right)\right] \\ &\quad \times \exp\left(-a^\dagger a\right) \exp\left[\alpha_0^* a \left(\ln\sqrt{\Gamma_3'}\right)\right] a^m : |z\rangle e^{-2\left(z\alpha^* - z^*\alpha\right)} \\ &= |\alpha_0|^{2n} e^{2|\alpha|^2 - |\alpha_0|^2} \int \frac{d^2 z}{\pi^2} (-1)^m z^{*m} z^m e^{-|z|^2 + Kz - Lz^*} \\ &= |\alpha_0|^{2n} e^{2|\alpha|^2 - |\alpha_0|^2} e^{-KL} H_{m,m}(K, L), \end{aligned} \quad (56)$$

where

$$K = \alpha_0^* \left(\ln\sqrt{\Gamma_3'}\right) - 2\alpha^*, \qquad L = \alpha_0 \left(\ln\sqrt{\Gamma_3'}\right) - 2\alpha. \quad (57)$$

Further,

$$\begin{aligned} W_c\left(\alpha, \alpha^*\right) &= \sqrt{\Gamma_3'} e^{(ig+\kappa)t} \sum_{m,n=0}^{\infty} \frac{(\Gamma_+')^m}{m!} \frac{(\Gamma_-')^n}{n!} W_{cmn}\left(\alpha, \alpha^*\right) \\ &= \sqrt{\Gamma_3'} e^{(ig+\kappa)t + \left(\Gamma_-' - 1\right)|\alpha_0|^2 + 2|\alpha|^2 - KL} \sum_{m=0}^{\infty} \frac{(\Gamma_+')^m}{m!} H_{m,m}(K, L) \\ &= \sqrt{\Gamma_3'} e^{(ig+\kappa)t + \left(\Gamma_-' - 1\right)|\alpha_0|^2 + 2|\alpha|^2 - KL} \sum_{m=0}^{\infty} \left(-\Gamma_+'\right)^m L_m(KL). \end{aligned} \quad (58)$$

Obviously, when $t = 0$, both (53) and (58) are just the Wigner function of the coherent state $|\alpha_0\rangle \langle\alpha_0|$, i.e.,

$$W_\pm\left(\alpha, \alpha^*\right)|_{t=0} = W_c\left(\alpha, \alpha^*\right)|_{t=0} = \frac{1}{\pi} e^{-2|\alpha_0 - \alpha|^2}. \quad (59)$$

In sum, we have adopted the entangled state approach for treating the time evolution of density operators in the Raman-coupled with cavity damping and their Wigner functions can also be derived for the given initial state. This is an alternate approach for conveniently tackling this task in a concise way, and provides us with a fresh view of the time-evolution of density operators as well.

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